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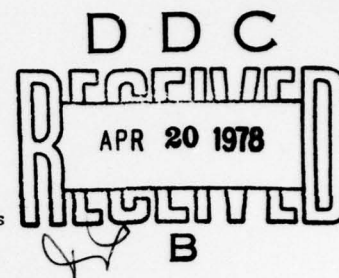
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ON MINIMUM ENERGY CONTROL OF COMMUTATIVE BILINEAR SYSTEMS\*

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## Abstract

Minimum energy control problems are considered for commutative bilinear systems with and without end point constraints. Optimal controls are shown to be constant vectors determined by the boundary conditions when the terminal state belongs to the reachable set. Sufficient conditions for uniqueness of solutions are derived for the minimum energy problem without a terminal constraint. Application to a missile intercept problem is discussed in which the pursuer possesses thrust modulation in addition to thrust vectoring.

## I. Introduction

Bilinear control systems have received increasing attention in recent years due, in part, to their natural applications in various engineering, biological and socio-economic systems, Mohler [1], and in part to their intrinsically nearly linear structure, Bruni et al [2], Brockett [3]. In addition, the study of bilinear control systems has potential applications to systems containing sinusoidal nonlinearities, Lo and Willsky [4], especially those arising in spatial flight mechanics. This observation in conjunction with a missile intercept problem motivated the study of the bilinear regulator problem discussed in this paper.

The focus of this paper is on "commutative" bilinear systems, i.e. the special class of bilinear control systems in which the coefficient matrices commute with one another. This class has been studied by Sussmann [5] relative to "bang-bang" control functions. Here, the minimum energy control of such commutative bilinear systems is investigated with and without end point constraints on the state. Concerning the regulator problem without end point constraints, it is shown in Section II that the optimal control is a constant

vector determined by the initial conditions. In the case of the regulator problem with end point constraint, it is first shown that a given terminal state  $x_1 = x(T)$  for a commutative bilinear system is constant reachable if and only if it is reachable by a time-dependent control. If  $x_1$  belongs to the reachable set, it is then shown that the optimal control is a constant vector determined by the boundary conditions. Uniqueness of solutions to the regulator problems of Section II is discussed in Section III. As an example of a bilinear system, a missile intercept problem is discussed in Section IV in which the pursuer possesses thrust modulation in addition to thrust vectoring. This additional degree of freedom for the pursuer facilitates the formulation of the problem as a regulator problem for a commutative bilinear system with end point constraint. It is then pointed out how a closed-form solution can be obtained for this example.

## II. Problem Statement and Existence of Solutions

Consider the multi-input bilinear system†

$$\dot{x} = \left( A + \sum_{i=1}^m B_i u_i \right) x, \quad x(t_0) = x_0 \in \mathbb{R}^n,$$

$$t \in [t_0, T], \quad u = u(t) = (u_1 \cdots u_m)' \quad (1)$$

where  $A, B_i, i=1 \cdots m$ , are  $n \times n$  constant matrices,  $u \in L^2([t_0, T], \mathbb{R}^m)$ , the class of  $m$ -vector valued square integrable functions on  $[t_0, T]$ .

Definition 1: The system (1) is called a commutative bilinear system if every pair of the matrices  $(A, B_1 \cdots B_m)$  commute with each other.

As mentioned earlier, the commutative bilinear system has been studied by Sussmann [5] in which it was shown that the attainable set is closed relative to "bang-bang" controls; Baras and Hartson [6]

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† A direct input term  $Cu$  can always be absorbed into the bilinear term  $\sum B_i u_i x$  by introducing an additional state,  $x_{n+1} = 1$ , to obtain Eq. (1).

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recently extended these results to delayed commutative bilinear systems. The problem considered here is the minimum energy control of a commutative bilinear system. Brockett [7] has obtained a solution to the minimum control energy problem with a fixed terminal state in the case of nxn matrix state commutative bilinear systems with  $\det X \neq 0$ . By contrast, it will be shown that this problem for the vector state system (1) has a simple solution which possesses an easily implemented character.

Two kinds of cost functions are investigated here,

$$J_1(u) = x'(T)Qx(T) + \int_{t_0}^T u'(t)Ru(t)dt \quad (2)$$

without a terminal constraint on the state, and

$$J_2(u) = \int_{t_0}^T u'(t)Ru(t)dt \quad \text{with } x(T) = x_1 \quad (3)$$

where  $x_1$  is a prespecified vector; Q and R are nxn nonnegative definite and positive definite symmetric constant matrices respectively, and prime denotes the matrix transpose operation.

In consideration of the existence of optimal controls to the above-mentioned problems, the reachable set plays a very important role. Therefore, we shall give the following definitions of reachable set and reachability of a bilinear system. With these definitions we can also reveal some interesting characteristics of the reachable set for a commutative bilinear system.

**Definition 2:** A set  $Z(x_0;U)$  is called a reachable set associated with (1) if  $Z(x_0;U) = \{x(T) \in R^n : x(t_0) = x_0, u(t) \in U, x(t) \text{ satisfies (1) in some finite interval } [t_0, T]\}$ . A set  $Z(D;U)$  is called a reachable zone associated with (1) if

$$Z(D;U) = \bigcup_{x_0 \in D} Z(x_0;U).$$

**Definition 3:** System (1) is reachable to  $x_1$  with respect to  $x_0$  and U if there is an input  $u(t) \in U$  which steers it from  $x_0$  to  $x_1$  at some finite time T. System (1) is constant reachable to  $x_1$  with respect to  $x_0$  if there is a constant input function  $u_c$  which steers it from  $x_0$  to  $x_1$  at some finite time T.

The following theorem states an interesting property regarding the reachability of commutative bilinear systems.

**Theorem 1:**

The commutative bilinear system (1) is reachable to  $x_1$  with respect to  $x_0$  and  $L^2([t_0, T], R^m)$

if and only if it is constant reachable to  $x_1$  with respect to  $x_0$ .

**Remark:**

Theorem 1 assures that if  $u(t) \in L^2([t_0, T], R^m)$  steers the commutative bilinear system from  $x_0$  to  $x_1$  at T, then there exists a constant input function  $u_c$  which can do the same job as well. This enables us to study the commutative bilinear system with a class of simple easily implemented input functions, namely, the constant input functions.

**Proof:**

The sufficiency part is true by definition. To show necessity, suppose  $u(t) \in L^2([t_0, T], R^m)$  steers (1) from  $x_0$  to  $x_1$  at some finite time T. Then, since each pair of  $(A, B_1 \cdots B_m)$  commutes with each other, the solution of (1) at time T can be expressed as

$$x(T) = \phi_A(T, t_0) \prod_{i=1}^m \phi_{B_i}(T, t_0) x_0 = x_1$$

where  $\phi_A(t, t_0)$  and  $\phi_{B_i}(t, t_0)$  denote the state transition matrices associated with A and  $B_i u_i(t)$ , respectively. Choosing  $u_c$  such that

$$u_{c_i} = \frac{1}{T-t_0} \int_{t_0}^T u_i(s)ds$$

it is easily seen that  $\phi_{B_i}(T, t_0) = \phi_{c_i}(T-t_0)$ , where  $\phi_{c_i}$  is the state transition matrix corresponding to  $B_i u_{c_i}$ . Thus

$$\phi_A(T, t_0) \prod_{i=1}^m \phi_{B_i}(T, t_0) x_0 = \phi_A(T, t_0) \prod_{i=1}^m \phi_{c_i}(T-t_0) x_0 = x_1$$

which verifies the theorem.

This result holds for a slightly more general bilinear system in which  $A = A(t)$  is time-varying provided  $(A(t), B_1 \cdots B_m)$  commute with one another. For this class of bilinear systems, the reachable zone is much easier to characterize because one need only consider a constant input  $u_c$  as a set of parameters, then compute the corresponding transition matrices  $\phi_{c_i}(T-t_0)$ ,  $i=1, \dots, m$ , which characterize the reachable zone of the given system (1) with a set D of initial conditions.

Define the set of attainability K of (1) and (2) by

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$$K = \{(x^0(u), x(T;u)) \in \mathbb{R}^{n+1} : x^0(u) = \int_{t_0}^T u'(t) R u(t) dt,$$

$$x(T;u) = \phi(T, t_0; u) x_0, u \in L^2([t_0, T], \mathbb{R}^m)\} \quad (4)$$

where  $\phi(t, t_0; u)$  is the state transition matrix of (1) for each given  $u$ . It is clear that  $K$  consists of the pairs of cost and terminal state corresponding to all admissible input functions as coordinates.

Now consider the map  $G: K \rightarrow [0, \infty)$  defined by

$$G(x^0(u), x(T;u)) = x'(T;u) Q x(T;u) + x^0(u). \quad (5)$$

Hence,  $J(u) = G(x^0(u), x(T;u))$ .

The following lemma reveals some characteristics of the set of attainability  $K$ .

**Lemma 1:** Every subset  $N_c(p)$  of  $K$  generated by constant controls on  $[t_0, T]$  is compact, where

$$N_c(p) = \{(x^0(u), x(T;u)) \in K : 0 \leq x^0(u) \leq p, p < \infty, \text{ and } \dot{u}(t) = 0\}. \quad (6)$$

**Proof:**

The proof is straightforward by the Euclidean topology and the linear structure of (1) for each given  $u$ . Since  $N_c(p) \subset K \subset \mathbb{R}^{n+1}$  and  $0 \leq x^0(u) \leq p$ , one can easily establish the closedness and boundedness of the set

$$E_0 = \{x(T;u) : (T-t_0)u'Ru \leq p\},$$

which proves the lemma.

The compactness of  $N_c(p)$  for each given positive  $p$  assures the existence of an optimal solution of the commutative bilinear system (1) and (2) as given in the next theorem.

**Theorem 2:**

Given a commutative bilinear system (1), the optimal controls which minimize the cost (2) are in the form of a constant vector  $u^*$  which satisfies the transcendental equation:

$$\begin{pmatrix} u_1^* \\ \vdots \\ u_m^* \end{pmatrix} = -\frac{1}{2} P^{-1} \begin{pmatrix} x_0' \phi_1'(T, t_0) \prod_{i=1}^m \phi_i'(T-t_0) B_1' Q \prod_{i=1}^m \phi_i(T-t_0) \phi_A(T, t_0) x_0 \\ \vdots \\ x_0' \phi_m'(T, t_0) \prod_{i=1}^m \phi_i'(T-t_0) B_m' Q \prod_{i=1}^m \phi_i(T-t_0) \phi_A(T, t_0) x_0 \end{pmatrix} \quad (7)$$

where  $\phi_i(T-t_0)$  and  $\phi_A(t, t_0)$  are the state transition matrices associated with  $u_i^* B_i$  and  $A$  respectively.

**Proof:**

Theorem 1 allows us to assume that for a given  $x(T, u)$ , there exists a constant vector  $u_c$  such that  $x(T, u_c)$  coincides with  $x(T, u)$  and

$$(T-t_0)u_c = \int_{t_0}^T u(t) dt. \quad (8)$$

Consider the costs associated with  $u$  and  $u_c$ , and assume without loss generality that  $R$  is a diagonal matrix with positive elements  $r_i, i=1 \dots m$ :

$$\int_{t_0}^T u_c' R u_c dt = \sum_{i=1}^m r_i u_{c_i}^2 (T-t_0) = U_1$$

$$\int_{t_0}^T u'(t) R u(t) dt = \sum_{i=1}^m r_i \int_{t_0}^T u_i^2(t) dt = U_2$$

From (8) and Hölder's inequality, we have

$$u_{c_i}^2 (T-t_0)^2 = \left( \int_{t_0}^T |u_{c_i}| dt \right)^2 \leq \left( \int_{t_0}^T |u_i(t)| dt \right)^2 \leq \int_{t_0}^T u_i^2(t) dt (T-t_0)$$

for  $i=1 \dots m$ . Hence  $U_1 \leq U_2$ . The equality is achieved if and only if  $u_i(t) = c$ , a constant.

Therefore, if the minimum energy exists, it must be incurred by a constant input  $u^*$ . But this is indeed the case because the function  $G(\cdot, \cdot)$  is continuous on  $N_c(p)$ , and  $N_c(p)$  is compact for any given  $p$ , so that  $J_1(u)$  attains its minimum on  $N_c(p)$ .

With the assurance that  $u^*$  exists, the Maximum Principle can be used to derive the characteri-

zation given by (7); i.e. along with an optimal trajectory for (1) and (2) the state, co-state and

control satisfy

$$\dot{x}^* = (A + \sum_{i=1}^m B_i u_i^*) x^*, \quad x(t_0) = x_0$$

$$\dot{p}^* = -(A' + \sum_{i=1}^m B_i' u_i^*) p^*, \quad p(T) = -Qx(T)$$

$$u^*(t) = \frac{1}{2} R^{-1} \begin{pmatrix} x^{*'} B_1' p^* \\ \vdots \\ x^{*'} B_m' p^* \end{pmatrix}$$

From these relations it can be verified directly that  $\frac{du^*(t)}{dt} = 0$ , using the commutativity assumption, so that putting  $t = T$  into  $u^*(t)$  and using  $x(T) = \phi_A(T, t_0) \prod_{i=1}^m \phi_{C_i}(T-t_0) x_0$  leads ultimately to expression (7). This proves Theorem 2.

Theorem 2 states the simple character of the minimum energy control problem for commutative bilinear systems, i.e. the optimal controls are simply constant vectors which satisfy the transcendental equation (7). This fact enables us to treat the optimal commutative bilinear system as a fixed linear system for which the explicit solution is immediately available. As in the case with Theorem 1, this result also holds for a slightly more general system in which  $A = A(t)$  is a  $n \times n$  time-varying matrix which commutes with  $B_i (i=1 \dots m)$  because in the proof the time-dependence of  $A$  does not play any role. To study the minimum energy problem associated with cost function (3), i.e. with a terminal constraint, we acknowledge that a bilinear system does not generally have the global controllability property as a linear system which has been quite thoroughly characterized. Therefore, we shall limit our attention only to the reachable zone  $Z$  associated with each bilinear system rather than the whole space  $R^n$  as the target set when we are dealing with the cost (3). In other words, the minimization of control energy is taken over the set  $U_c$  of admissible controls which consist of those constant input functions which do steer the given system to the desired target set at a certain finite time.

The following theorem assures the existence of a constant optimal control to the minimum energy problem (1) and (3).

#### Theorem 3:

Given a commutative bilinear system (1) with the cost (3). If  $x_1$  belongs to the reachable set  $Z(x_0; L^2([t_0, T], R^m))$ , then there exists a constant optimal control  $u_c^*$  which steers (1)

from  $x_0$  to  $x_1$  at  $T$  and minimizes the associated cost (3). Furthermore,  $u_c^*$  satisfies the transcendental equation:

$$x_1 = \phi_A(T, t_0) \prod_{i=1}^m \phi_{C_i}(T-t_0) x_0 \quad (9)$$

#### Proof:

Suppose

$$U = \{u(t) \in L^2([t_0, T], R^m) : \phi_A(T, t_0) \prod_{i=1}^m \phi_{C_i}(T, t_0) x_0 = x_1\}$$

is the set of admissible controls. Then  $U$  is non-empty by hypothesis, and from Theorem 1 there exists a non-empty subset  $E$  consisting of all elements of  $U$  which are constant input functions. For each  $u(t) \in U$ , there is a  $u_c \in U$  such that

$$\int_{t_0}^T u(t) dt = \int_{t_0}^T u_c dt.$$

Comparing the costs associated with  $u(t)$  and  $u_c$  and using a similar argument as that in Theorem 2, one can show that the minimum energy is incurred by a constant input  $u_c^* \in U_c$ . Equation (9) then follows by a direct computation of the solution of the 'linear' fixed system (1). Q.E.D.

In order to obtain  $u^*$  from equation (7) or (9) by iterative schemes, it is interesting to study the uniqueness property of the solution to these equations. Next, we will use the property of monotonically increasing maps to show the uniqueness of solutions to (7) for a class of bilinear systems.

### III. Uniqueness of the Minimum Energy Control

#### Definition 4:

A continuous map  $G$  from  $R^m$  into itself is monotonically increasing if  $\langle x_1 - x_2, G(x_1) - G(x_2) \rangle \geq 0$  for all  $x_1$  and  $x_2$  in  $R^m$ , where  $\langle \cdot, \cdot \rangle$  is the usual inner product, i.e.  $\langle x, y \rangle = x'y$ .

The following lemma states the uniqueness property associated with a monotonic map. The proof can be found in Minty [8].

**Lemma 2:** Given a monotonically increasing map  $G$ , then the solution of the equation  $x + G(x) = 0$ ,  $x \in R^m$ , is unique.

Based on this lemma, sufficient conditions for uniqueness can be derived as summarized in

the following theorem.

**Theorem 4:**

There is a unique optimal solution to the commutative bilinear system (1) with cost (2) if the matrix  $R^{-1}Z(v_0)$  is non-negative definite for all  $v_0$  in  $R^n$ , where  $Z(v_0) = (Z_{ij})$  and

$$Z_{ij} = v_0'(B_j^1 B_1^1 Q + B_i^1 B_j^1 Q) v_0, \quad i, j = 1, 2, \dots, m. \quad (10)$$

**Proof:**

We define  $G(u^*) = \frac{1}{2} R^{-1} g$  where

$$g_j = y_0' \phi_A'(T, t_0) \prod_{i=1}^m \phi_i'(T-t_0) B_i^1 Q \prod_{i=1}^m \phi_i(T-t_0) \phi_A(T, t_0) y_0, \\ j = 1, \dots, m$$

and compute the Fréchet differential of  $G$  at  $u^*$  with increment  $h$ :

$$dG(u^*; h) = \left[ \frac{1}{2} R^{-1} Z(y_0) \right] \cdot h, \quad h \in R^m$$

where

$$Z_{ij} = y_0' \phi_A'(T, t_0) \prod_{k=1}^m \phi_k'(T-t_0) [B_j^1 B_1^1 Q + B_i^1 B_j^1 Q] \\ \times \prod_{k=1}^m \phi_k(T-t_0) \phi_A(T, t_0) y_0.$$

By hypothesis,  $R^{-1}Z(v_0) \geq 0$  which implies that  $G$  is monotonically increasing.<sup>†</sup> Therefore, Lemma 2 gives the uniqueness result. Q.E.D.

In the case of a single-input bilinear system, condition (10) simplifies significantly to the non-negative definiteness of the matrix  $B_1^2 Q + B_1^1 Q B_1$ .

As far as the minimum energy problem associated with a terminal constraint is concerned, sufficient conditions for the uniqueness of the solutions to Eq. (9) are more difficult to derive because of the nonuniqueness of solutions to the TPBVP associated with a nonlinear system.

The following simple example illustrates the nonuniqueness of optimal controls for a

minimum energy problem with a fixed terminal constraint.

**Example 1:**

Given the bilinear system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} u,$$

$$x(0) = (0, 1)'$$

with the cost to be minimized and terminal constraint given by

$$J(u) = \int_0^1 u^2(t) dt, \quad x(1) = \begin{bmatrix} 0 \\ -e \end{bmatrix}$$

It can be easily seen that  $(A, B_1)$  commute; hence, by Theorem 3 an optimal control  $u_c^*$  satisfies the following equation:

$$\begin{bmatrix} 0 \\ -e \end{bmatrix} = e \begin{bmatrix} \cos u_c^* & \sin u_c^* \\ -\sin u_c^* & \cos u_c^* \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e \sin u_c^* \\ e \cos u_c^* \end{bmatrix}.$$

Solving for  $u_c^*$ , we obtain  $u_c^* = k\pi$ ,  $k = \pm 1, \pm 3, \dots$

In order to have the total control energy minimized, the integers with the smallest absolute value are chosen, i.e.  $k_1 = 1$ ,  $k_2 = -1$ . Clearly,  $u_1^* = \pi$  and  $u_2^* = -\pi$  both incur a minimal cost of  $\pi^2$  and steer the given system to the desired final state. Hence both  $u_1^*$  and  $u_2^*$  are optimal controls.

The lack of uniqueness of the minimum energy control problem for a commutative system associated with a terminal constraint is, in general, expected because the nonlinear two-point boundary value problem generally does not have a unique solution which prevents the uniqueness of optimal solutions in many cases.

**IV. A Two-Dimensional Missile Intercept System**

It is assumed that for a typical high-speed pursuing missile and short initial range, the maneuvering of the vehicles can be restricted to a

<sup>†</sup> Because  $R^{-1}Z(v_0) \geq 0$  implies that the Fréchet derivative of  $G$  is monotonically increasing, which implies the monotonicity of  $G$ , see Vainberg [9].



two-dimensional plane. Choose the coordinate system fixed in the missile as shown in Figure 1. Denote the angular rate of the missile and the target with respect to a non-rotating reference frame as  $u_p$  and  $u_T$ , respectively.

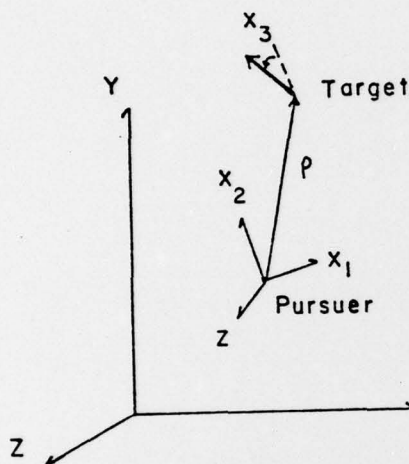


FIGURE 1

The equations of motion are then described by [10]

$$\begin{aligned}\dot{x}_1 &= -v_T \sin x_3 + x_2 u_p \\ \dot{x}_2 &= v_T \cos x_3 - x_1 u_p - v_p \\ \dot{x}_3 &= u_T - u_p\end{aligned}\quad (11)$$

where  $v_T$  and  $v_p$  are the line speeds of the target and the missile relative to air;  $x_1$  and  $x_2$  are the horizontal and vertical distance from the missile, and  $x_3$  is the relative angle between the headings of the missile and target measured counterclockwise.

The system (11) can be transformed into a homogeneous bilinear system by introducing three auxiliary states:  $x_4 = \sin x_3$ ,  $x_5 = \cos x_3$  and  $x_6 = 1$ . That is,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\quad (12)$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & -v_p \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\quad (13)$$

$\mathbf{x}(t_0)$

$$= (x_1(t_0), x_2(t_0), x_3(t_0), \sin x_3(t_0), \cos x_3(t_0), 1)'$$

in which  $u = u_p$  is defined as the control variable.

The objective is to find a square integrable function  $u^*(t)$  which steers the missile to the target at some finite time  $T$  (a free time formulation) while the total control energy consumed over this interval  $[t_0, T]$  is minimized. The following performance index is thus considered:

$$J(u) = \int_{t_0}^T u^2(t) dt, \quad T > t_0 \quad (14)$$

subject to

$$x_1(T) = x_2(T) = 0, \quad (15)$$

In addition to the usual thrust vectoring, the missile is assumed to possess thrust modulation capabilities so that  $v_p(t)$  can be adjusted in addition to  $u_p(t)$ . Rather than regard  $v_p$  as an independent control, a proportionality relationship between  $u_p(t)$  and  $v_p(t)$  is postulated:

$$v_p = \gamma u_p \quad (16)$$

with the proportionality parameter  $\gamma$  to be determined by the boundary conditions as indicated presently. With this postulated relation, Eq. (13) becomes

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & 0 \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\quad (17)$$



and it can be readily verified that  $A$  and  $B$  commute for all  $\gamma$ . Consequently, in the event that  $v_T$  and  $u_T$  are constants, the solution of the system (12) and (17) can be expressed analytically.

We will first resolve the terminal constraint problem by considering the intercept angle as a parameter, then incorporate the solutions with the minimum energy problem. Consideration should be given to two separate cases in which  $u_T$  is zero and non-zero.

From Equation (11), denote the intercept angle by  $\beta$ , i.e.

$$\beta = x_3(T) = x_3(t_0) - \int_{t_0}^T u(s) ds + u_T(T-t_0). \quad (18)$$

The terminal constraint (15) becomes<sup>+</sup>

$$0 = \begin{bmatrix} [x_1(t_0) + \gamma] \cos\{u_T(T-t_0) + x_3(t_0) - \beta\} \\ -[x_1(t_0) + \gamma] \sin\{u_T(T-t_0) + x_3(t_0) - \beta\} \\ x_2(t_0) \sin\{u_T(T-t_0) + x_3(t_0) - \beta\} - \gamma \\ x_2(t_0) \cos\{u_T(T-t_0) + x_3(t_0) - \beta\} \end{bmatrix} \quad (19)$$

$$+ \frac{v_T}{u_T} \begin{bmatrix} \cos \beta - \cos[u_T(T-t_0) - \beta] \\ \sin[u_T(T-t_0) - \beta] + \sin \beta \end{bmatrix}$$

In other words, the terminal constraint problem of the differential system (12) and (17) has been reduced to solving a pair of nonlinear algebraic equations (19) of transcendental type for an appropriate set  $(\gamma, \beta, T)$ . A solution often exists for this case in which the number of unknowns exceeds the number of equations.

The next proposition shows the existence of a triple  $(\gamma, \beta, T)$  which solves (19) for different initial conditions  $(x_1(t_0), x_2(t_0), x_3(t_0))$  in  $R^3$ , the analytic expression for this triple, as well as the proof of the proposition, is given in Wei, [11] and [12].

**Proposition 1:** (a) If  $u_T$  is a non-zero constant, then there exists a triple  $(\gamma, \beta, T)$  satisfying (18) and (19) which solves the terminal constraint problem (15) for every  $(x_1(t_0), x_2(t_0), x_3(t_0)) \in R^3$ .

<sup>+</sup>The usual limit will be taken when  $u_T$  approaches zero.

(b) If  $u_T$  is zero, then there exists a triple  $(\gamma, \beta, T)$  which solves the same terminal constraint problem (15) for every  $(x_1(t_0), x_2(t_0), x_3(t_0)) \in R^3 \setminus E^+$ , where

$$E = \{(0, y, z) \in R^3: \text{Either } y > 0 \text{ and } z = (2k+1)\pi, \text{ or } y < 0 \text{ and } z = 2k\pi; k = 0, \pm 1, \dots\}.$$

Since the system (12) and (17) is a commutative bilinear system, the results in section II are applicable to the minimum energy problem (14) and (15). The set  $U_c$  of admissible controls in this case includes those input functions  $u(t)$  satisfying the algebraic equations (18) and (19). The next proposition which gives the explicit form of the optimal controls to the problem (14) and (15) is a direct consequence of Theorem 3.

**Proposition 2:** Given the system (12) and (17), there exists an optimal control  $u^* \in U_c$  which minimizes the cost (14) and steers the system to  $x_1(T) = x_2(T) = 0$  at some  $T > t_0$  for each appropriate set of initial conditions  $(x_1(t_0), x_2(t_0), x_3(t_0), u_T, v_T)$ . This control is given by

$$u^*(t) = u_T + \frac{x_3(t_0) - \beta}{T - t_0} \quad (20)$$

where  $T$  and  $\beta$  are given as discussed in Proposition 1.

**Proof:**

Theorem 3 implies the existence of constant optimal controls  $u^* \in U_c$ . By equation (18),  $u^*$  is given as in (20).

The striking character of this optimal control law in a constant form is not completely without expectation because the control aspects proposed here include two channels. One is the angular maneuver of the missile which counterbalances (offsets) the angular maneuver of the target, while the additional degree of freedom introduced by  $\gamma$  carries out the major pursuit part of the problem and leads to a simple solution having some intuitive sense.

On the other hand, it should be noted that although an optimal control is in a constant form (a step function), a sub-optimal control law can always be constructed which will drive the missile to the target at some  $T$ , with an appropriate intercept angle  $\beta$  and for some ratio  $\gamma$ , as long as the area swept out satisfies (19). This allows the control engineer a great deal of flexibility in the design of a feasible easily implemented sub-optimal controller.

<sup>+</sup> $A \setminus B$  denotes the complement of  $B$  in  $A$ .

## V. Conclusions

It has been shown that the optimal controls are in the form of constant vectors determined by the boundary conditions for a class of minimum energy control problems associated with commutative bilinear systems. Sufficient conditions for the uniqueness of solutions were obtained for the minimum energy problem without a terminal constraint on the state. In the case of the control problem with a fixed terminal constraint belonging to the reachable set, it was shown that a variety of different controls can be applied to reach the desired terminal state provided that all such controls satisfy an area condition. This allows greater flexibility from a design point of view and has been exploited in [12] for the singularly perturbed commutative bilinear system.

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